

Shortest Distance in Modular Hyperbola and Least Quadratic Nonresidue

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Abstract

In this paper, we study how small a box contains at least two points from a modular hyperbola $xy \equiv c \pmod{p}$. There are two such points in a square of side length $p^{1/4+\epsilon}$. Furthermore, it turns out that either there are two such points in a square of side length $p^{1/6+\epsilon}$ or the least quadratic nonresidue is less than $p^{1/(6\sqrt{e})+\epsilon}$.

1 Introduction and Main results

Let $p > 2$ be a prime and $(c, p) = 1$. We consider the modular hyperbola

$$H_c := \{(x, y) : xy \equiv c \pmod{p}\}.$$

We are interested in the shortest distance between two points in H_c . Rather than distances, we consider how small a box

$$B(X, Y; H) := \{(x, y) : X + 1 \leq x \leq X + H \pmod{p}, Y + 1 \leq y \leq Y + H \pmod{p}\}$$

contains two points in H_c where X and Y run over $0, 1, \dots, p-1$.

By Hölder's inequality and Weil's bound on character sum, we have

Theorem 1 *For all $(c, p) = 1$, there exist some $0 \leq X, Y \leq p-1$ so that*

$$|H_c \cap B(X, Y; H)| \geq 2$$

if $H \gg_\epsilon p^{1/4+\epsilon}$.

Now let us switch to the subject of the least quadratic nonresidue modulo p . Many people have been interested in the upper bound for n_p . By Polya-Vinogradov bound on character sums, we have

$$n_p \ll p^{1/2} \log p.$$

Vinogradov [5] applied a trick and got

$$n_p \ll_\epsilon p^{1/(2\sqrt{e})+\epsilon}.$$

Burgess [1] proved a new bound on short character sum which together with Vinogradov's trick yielded

$$n_p \ll_\epsilon p^{1/(4\sqrt{e})+\epsilon}.$$

Recall a recent result of Heath-Brown [2] and Shao [3] on mean-value estimates of character sums:

Theorem 2 Given $H \leq p$, a positive integer and any $\epsilon > 0$. Suppose that $0 \leq N_1 < N_2 < \dots < N_J < p$ are integers satisfying $N_{j+1} - N_j \geq H$ for $1 \leq j < J$. Then

$$\sum_{j=1}^J \max_{h \leq H} |S(N_j; h)|^{2r} \ll_{\epsilon, r} H^{2r-2} p^{1/2+1/(2r)+\epsilon}$$

where

$$S(N; H) := \sum_{N < n \leq N+H} \chi(n).$$

and χ is any non-principal character modulo p .

Applying the above theorem, we can show that

Theorem 3 For any $\epsilon > 0$, we have either

$$n_p \ll_{\epsilon} p^{1/(6\sqrt{\epsilon})+\epsilon}$$

or, for any $(c, p) = 1$ and $H \gg_{\epsilon} p^{1/6+\epsilon}$,

$$|H_c \cap B(X, Y; H)| \geq 2$$

for some $0 \leq X, Y \leq p-1$.

It is probably the case that the above two statements are true simultaneously. The paper is organized as follows. In section 2, we give a basic argument transforming the existence of two close points in the modular hyperbola to a certain equality in Legendre symbol. Then we prove Theorem 1 in section 3 and Theorem 3 in section 4.

Some Notations Throughout the paper, p stands for a prime. The symbol $|S|$ denotes the number of elements in the set S . We also use the Legendre symbol $(\frac{\cdot}{p})$. The notation $f(x) = o(g(x))$ means that the ratio $f(x)/g(x)$ is going to zero as $x, p \rightarrow \infty$. The notations $f(x) \ll g(x)$, $g(x) \gg f(x)$ and $f(x) = O(g(x))$ are equivalent to $|f(x)| \leq Cg(x)$ for some constant $C > 0$. Finally, $f(x) \ll_{\lambda_1, \dots, \lambda_k} g(x)$, $g(x) \gg_{\lambda_1, \dots, \lambda_k} f(x)$ and $f(x) = O_{\lambda_1, \dots, \lambda_k}(g(x))$ mean that the implicit constant C may depend on $\lambda_1, \dots, \lambda_k$.

2 The Basic Argument

For $(c, p) = 1$, suppose $|H_c \cap B(X, Y; H)| \geq 2$ for some $0 \leq X, Y \leq p-1$. This means that

$$xy \equiv c \pmod{p}, \text{ and } (x+a)(y+b) \equiv c \pmod{p} \quad (1)$$

for some $1 \leq x, y \leq p-1$ and $1 \leq a, b \leq H$. After some algebra, one can show that (1) is equivalent to

$$bx + ac\bar{x} + ab \equiv 0 \pmod{p}$$

where \bar{x} stands for the multiplicative inverse of x modulo p (i.e. $x\bar{x} \equiv 1 \pmod{p}$). This, in turn, is equivalent to

$$(2bx + ab)^2 \equiv (ab)^2 - 4abc \pmod{p}.$$

Therefore $|H_c \cap B(X, Y; H)| \geq 2$ if and only if

$$\left(\frac{ab}{p}\right) \left(\frac{ab-4c}{p}\right) = 1$$

for some $1 \leq a, b \leq H$. We are going to restrict our attention to even $a = 2a'$'s and $b = 2b'$'s. So we want

$$\left(\frac{a'b'}{p}\right) \left(\frac{a'b'-c}{p}\right) = 1 \text{ for some } 1 \leq a', b' \leq H/2. \quad (2)$$

3 Proof of Theorem 1

Throughout this section, we assume that $H \gg_\epsilon p^{1/4+\epsilon}$. We want to show that

$$\sum_{a' \leq H/2} \sum_{b' \leq H/2} \left(\frac{a'b'}{p} \right) \left(\frac{a'b' - c}{p} \right) = o(H^2).$$

Then either we have two pairs with $(\frac{a'_1 b'_1 - c}{p}) = 0 = (\frac{a'_2 b'_2 - c}{p})$ which gives Theorem 1 automatically; or at most one such pair equal to 0 which would imply that $(\frac{a'_1 b'_1}{p})(\frac{a'_1 b'_1 - c}{p}) = 1$ and $(\frac{a'_2 b'_2}{p})(\frac{a'_2 b'_2 - c}{p}) = -1$ for some $1 \leq a'_1, b'_1, a'_2, b'_2 \leq H/2$. However, we are going to restrict a' to a special form, namely $a' = uv$ with $1 \leq u \leq U$, $1 \leq v \leq V$ and $UV = H/2$. So let us consider

$$S := \sum_{u \leq U} \sum_{v \leq V} \sum_{b' \leq H/2} \left(\frac{uvb'}{p} \right) \left(\frac{uvb' - c}{p} \right).$$

Then

$$S \leq \sum_{v \leq V} \sum_{b' \leq H} \left| \sum_{u \leq U} \left(\frac{u}{p} \right) \left(\frac{uvb' - c}{p} \right) \right| \ll_\epsilon p^\epsilon \sum_{n \leq VH} \left| \sum_{u \leq U} \left(\frac{u}{p} \right) \left(\frac{un - c}{p} \right) \right|$$

as the number of divisors of n is $O_\epsilon(n^\epsilon)$. Now apply Hölder's inequality and get

$$\begin{aligned} S &\ll_{\epsilon, r} p^{\epsilon/(2r)} \left(\sum_{n \leq VH} 1 \right)^{(2r-1)/(2r)} \left(\sum_{n \leq p-1} \left| \sum_{u \leq U} \left(\frac{u}{p} \right) \left(\frac{un - 4c}{p} \right) \right|^{2r} \right)^{1/(2r)} \\ &\ll_{\epsilon, r} p^{\epsilon/(2r)} (VH)^{1-1/(2r)} (U^r p + U^{2r} p^{1/2})^{1/(2r)} \end{aligned}$$

by Lemma 4 in [4] which follows from Weil's bound on multiplicative character sums. Now we take $U = p^{1/(2r)}$ and $V = H/U$ with $1/r < \epsilon$. Then one can verify that $S = o(H^2)$ which implies that there is some $uv \leq H/2$ and $b' \leq H/2$ such that $(\frac{uvb'}{p})(\frac{uvb' - c}{p}) = 1$. This together with the argument in section 2 gives Theorem 1.

4 Proof of Theorem 3

Throughout this section, we assume that $H \gg_\epsilon p^{1/6+\epsilon}$. Suppose, for all $(c, p) = 1$,

$$\left(\frac{a'b'}{p} \right) \left(\frac{a'b' - c}{p} \right) = 1$$

for some $1 \leq a', b' \leq H/2$. This together with section 2 implies that

$$|H_c \cap B(X, Y; H)| \geq 2$$

for any $(c, p) = 1$ and $H \gg_\epsilon p^{1/6+\epsilon}$.

Now, suppose this is not the case. Then, for some $(c, p) = 1$,

$$\left(\frac{a'b'}{p} \right) \left(\frac{a'b' - c}{p} \right) = 0 \text{ or } -1$$

for all $1 \leq a', b' \leq H/2$. Suppose two such pairs give

$$\left(\frac{a'_1 b'_1 - c}{p} \right) = 0 = \left(\frac{a'_2 b'_2 - c}{p} \right).$$

This implies $|H_c \cap B(X, Y; H)| \geq 2$ automatically by section 2. Subsequently, we assume that

$$\left(\frac{a'b'}{p}\right)\left(\frac{a'b' - c}{p}\right) = -1$$

for all but at most one pair of $1 \leq a', b' \leq H/2$. This implies that

$$\left(\frac{a' - c\overline{b'}}{p}\right) = -\left(\frac{a'}{p}\right)$$

for all but at most one pair of $1 \leq a', b' \leq H/2$. Consequently,

$$\sum_{b' \leq H/2} \left| \sum_{a' \leq H/2} \left(\frac{a' - c\overline{b'}}{p}\right) \right|^{2r} \geq ([H/2] - 1) \left| \sum_{a' \leq H} \left(\frac{a'}{p}\right) \right|^{2r} =: ([H/2] - 1) |\Sigma|^{2r}.$$

Now we apply Theorem 2 with $N_{b'} = -c\overline{b'}$. First we claim that $c\overline{b'_1} - c\overline{b'_2}$ cannot be congruent to some $l \leq H$ modulo p for $1 \leq b'_1 < b'_2 \leq H/2$. For otherwise

$$c\overline{b'_1} - c\overline{b'_2} \equiv l \pmod{p}$$

for some $1 \leq l \leq H$. Let $a'_1 \equiv c\overline{b'_1} \pmod{p}$ and $a'_2 \equiv c\overline{b'_2} \pmod{p}$. Then (a'_1, b'_1) and (a'_2, b'_2) would be two points of the modular hyperbola H_c lying in a square of side length H which contradicts our assumption that no such square contains two such points. Therefore, we can apply Theorem 2 and get

$$([H/2] - 1) |\Sigma|^{2r} = \sum_{b' \leq H/2} \left| \sum_{a' \leq H/2} \left(\frac{a' - c\overline{b'}}{p}\right) \right|^{2r} \ll_{\epsilon, r} H^{2r-2} p^{1/2+1/(2r)+\epsilon}.$$

This implies that

$$\Sigma \ll_{\epsilon, r} H^{1-3/(2r)} p^{(r+1)/(4r^2)+\epsilon/(2r)} = o(H)$$

if r is sufficiently large. Hence we have that the character sum

$$\sum_{a' \leq H/2} \left(\frac{a'}{p}\right) = o(H).$$

Feeding this character sum estimate into the standard Vinogradov's trick in obtaining upper bound for the least quadratic nonresidue, we have the first half of Theorem 3.

References

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